Using DGS for working on realistic Billiards Tasks

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Abstract

In this contribution we first clarify what we mean by “realistic” billiards tasks, i.e. the mathematical model we work with. We then classify the interesting tasks that come up in the model and finally we provide some examples on how a DGS can be used for performing the respective tasks.

1. Introduction

So-called “Mathematical Billiards” using a variety of table geometries has been a topic of several didactical investigations (cf. Barabash 2003, Bettinaglio/Lehmann 1998, Shultz/Shiflett 1988), and there are numerous web pages where animations produced with dynamic geometry systems (DGS) can be viewed. Although mathematical billiards leads to a variety of interesting mathematical questions, it lacks somewhat authenticity since it obviously does not model the real situation (e.g. collision between balls). In our approach, a more realistic model called “Physical Billiards” is considered where balls are modelled as circles and loss of energy at collisions can be included. This model and its geometrical consequences for the behaviour of balls are outlined in the next section. We then set up a classification of tasks for this scenario.
In the main part of this paper we investigate how a DGS (Cinderella®) can be used for solving typical tasks in Carom Billiards providing several examples. The constructions should be flexible such that the positions of balls can be changed using the dragging mode. Some of the tasks can be solved directly by constructing a suitable direction for hitting the cue ball, for others so-called “soft constructions” (cf. Healy 2000) can be set up which can be used to find an approximate solution in the dragging mode. The latter constructions can also be interpreted as blue prints for real mechanisms including rotational and translational joints, and such mechanisms can really be built. For this, we also give an example. The billiards tasks are particularly suitable for student projects.

2. Modelling of Billiards

In order to have a realistic model of the behaviour of billiards balls, the balls are modelled as circles. This allows to consider the collision of two balls which is meaningless when balls are points as in mathematical billiards. Having circles already leads to a lot of interesting geometric questions as we will see in section 4. Obviously, balls do not run forever, so a realistic model has to take friction into account. Since geometric methods as are offered by a DGS only help in constructing directions of motions but not the motion itself, we do not consider this in the example section. Another observation which is not so obvious is concerned with the loss of kinetic energy when it comes to a collision. The collision (ball/cushion or ball/ball) is governed by two fundamental physical laws. In case of total elasticity, the motion can be computed using the conservation of momentum and the conservation of energy. If there is a loss of kinetic energy (i.e. kinetic energy is partially transformed into other forms of energy), then this has geometric consequences for determining motion directions. Without going into details concerning the algebraic side (for this compare Alpers (2006) or books on the physics of collision) we just consider these geometric consequences. Figure 1 shows what happens when a ball collides with a cushion and there is a loss of kinetic energy: There no longer is an equality relationship between the ingoing angle and the outgoing angle. If one splits up the velocity of the first ball into two components (one parallel, the other one orthogonal to the cushion), then the parallel one is not changed by the collision with the cushion whereas the orthogonal one not only reverses its direction but also has a smaller absolute value. The diminishing factor is called coefficient of restitution. It depends on the materials of the colliding bodies. So, the observable consequence of loss of kinetic energy is that the outgoing angle is larger than the incoming angle and this was actually observed in the Billiard machine we use in our laboratory (see Figure 3). There, the coefficient of restitution k is about 0.6. From Figure 1 is follows immediately that the relationship between ingoing and outgoing angle is given by $\tan\alpha = k \cdot \tan\bar{\alpha}$.

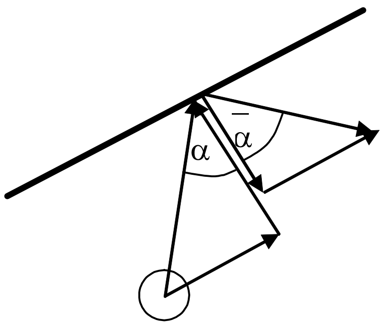

Figure 1

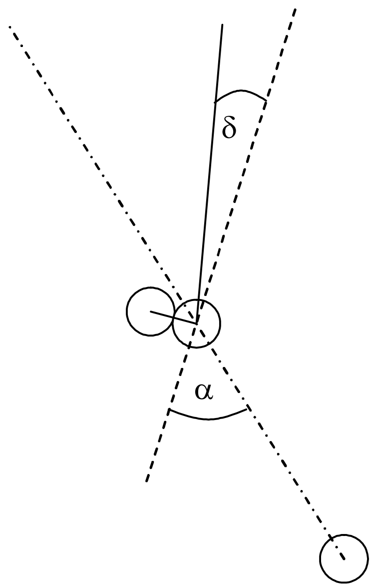

Figure 2

Loss of kinetic energy also has consequences when two balls collide. In the situation without loss of energy, the connection between the centre of the first ball and the center of the second ball at collision time is orthogonal to the motion direction of the first ball after collision (the dashed line in figure 2) since the velocity component "in the direction" of the second ball is totally transferred to the second ball. This is no longer the case when there is a loss of kinetic energy. An algebraic computation (cf. Alpers 2006) shows that there is a "deviation angle δ and the relation between the ingoing angle α and the deviation δ is again given by a tangens relationship: $\tan\delta = \frac{1}{2} \cdot (1-k)\tan\alpha$. When making constructions using a DGS that adhere to the

model including loss of kinetic energy, one has to take into account these relationships. How this can be done constructively will be shown in section 4.
In our model of physical billiards, the situation of the billiards machine is adequately modelled. For the human Billiards player, English shots (with spin) also have to be taken into account. But this would make the model too complex, and a DGS could not no longer help in constructing solutions (for such more advanced models cf. (Marlow 1995) and (Shepard 1997)).

Figure 3.

3. Tasks

Within the model described in the previous section there is a variety of interesting tasks. One can (and possibly should) start with a simple model just taking into account that balls are circles and then go on including more advanced features like friction or loss of kinetic energy. One can also start with situations involving just one or two balls and one (or none) cushion and then proceed to more complicated situations. The tasks can be roughly divided into two classes:

- When a certain situation is given (position of ball(s), velocity vector of ball(s)), what happens?
- When a certain position of the balls is given, determine the velocity vector of the cue ball such that a carambolage is achieved (maybe with further restrictions e.g. concerning the number of cushions that have to be touched)?

When a solution to one of the above tasks has been achieved, it should be easily adaptable to other situations of the same kind. Here, the usage of DGS is important since the dragging mode allows such easy modification if the respective construction was done properly.

4. Examples of DGS Usage

In this section we show how some example tasks can be solved using a DGS. We used the DGS Cinderella ® for this purpose but the operations could be performed similarly in any other well known DGS (the Cinderella files can be obtained from the author).

Example 1: Table construction

Even when right at the beginning just the Billiard table is to be constructed, the requirement for easy modification makes the task interesting. If one just draws a rectangle and then drags one point, the rectangular property is destroyed as can be seen in figure 4 (left part). In order to preserve this property one has to use parallelism (right part).

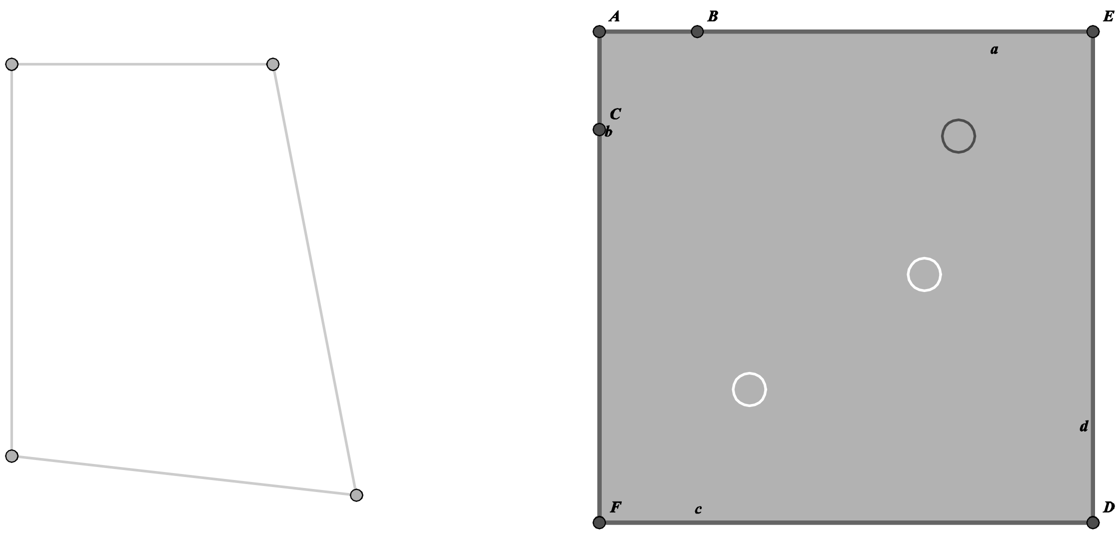

Figure 4

Example 2: Ball hits ball via cushion

In this example, a ball should hit another ball via a cushion centrically straight (and not just touch it). This is a task of the second kind as mentioned in the previous section. When one works in a model without loss of kinetic energy, then it is well known from mathematical billiards that a simple reflection is sufficient to find the velocity direction of the cue ball. When balls are modelled as circles one has to be a bit more careful since the reflection line is not the cushion but a parallel line at a distance of the ball radius (for the sake of brevity we will call this line the "virtual cushion" in the sequel of this paper). We consider here the more interesting model where loss of kinetic energy occurs and the relationship between ingoing and outgoing angle is governed by the tangens equation given in section 2. Figure 5 shows how this tangens equation can be guaranteed constructively by combining a reflection and a dilation by the factor 1/k where k is the coefficient of restitution. So, the centre of the second ball is to be reflected at the virtual cushion and then dilated by the factor 1/k. Multiplications can be performed geometrically by using intercept theorems as is shown in figure 5 (this should be known by the learner). For this, the coefficient of restitution must be available as a length (constructability questions for numbers are not discussed here). Figure 6 depicts the respective construction in Cinderella®.

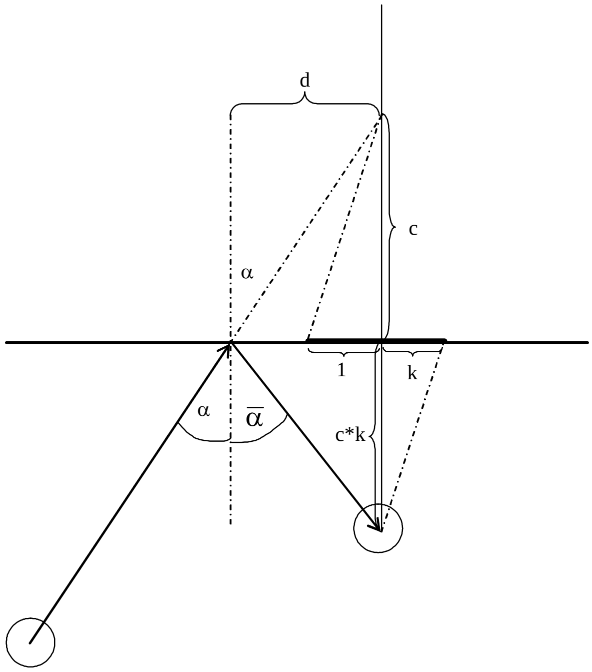

Figure 5

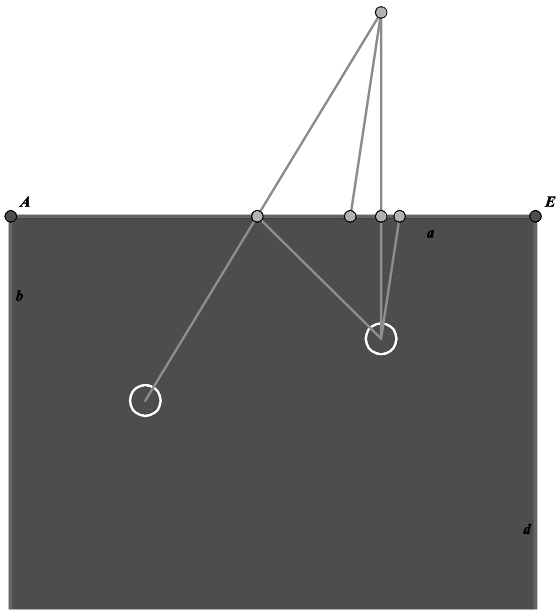

Figure 6

Healy (2000) calls such a construction a “hard” construction since the desired points are constructed directly without any dragging. One can also look for so-called “soft” constructions where one of the conditions is violated, and then use the dragging mode of a DGS to find an approximate solution. From the configuration in the approximate solution it might be possible to find additional properties which enable one to find a hard construction. Soft constructions are well known from “classical” problems like the trisection of an angle where approximate solutions can be obtained using a moving ruler.

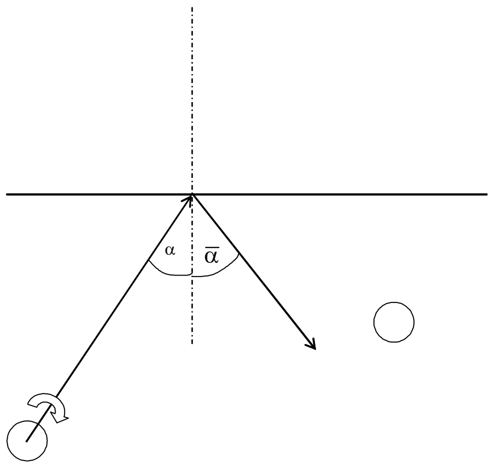

Figure 7

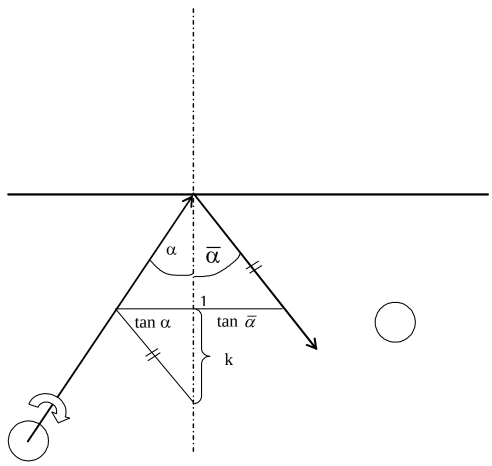

Figure 8

In our setting, one could omit the condition that the “reflected” ball hits the second ball and just consider the motion of the first ball for any velocity direction. This is shown in figure 7. When one then rotates the velocity direction one can get an approximate solution when the “outgoing line” contains the center of the second ball. This construction belongs to the first class of tasks mentioned in the previous section. Generally, the first class of tasks can produce soft constructions for getting approximate solutions for tasks belonging to the second class as will be seen in other examples below. It remains to be investigated whether the soft construction provides any hints for the hard construction in our example. Again, the relationship between ingoing and outgoing angle is determined by the tangens equation given in section 2, so one has to construct an angle whose tangens is a multiple of the tangens of a given angle. This leads to the main part of solving the hard construction, namely the geometric construction of a multiplication. In this sense, setting up the soft construction is helpful in that it leads to the essential point without taking into account the second ball and its properties. Moreover, it might lead (presumably with a little help by the instructor) to constructing the tangens values as distances as is done in figure 8. The tangens equation gives $\tan\alpha : \tan\overline{\alpha} = k : 1$ and this might give reason to look for intercept constructions. Having used the intercept theorem in the soft construction could be an incentive to look for similar applications (now involving the distance between the second ball and the virtual cushion) in the search for a hard construction. Figure 9 (Cinderella) shows the respecting soft construction which leads to an approximate solution when rotating the ingoing line.

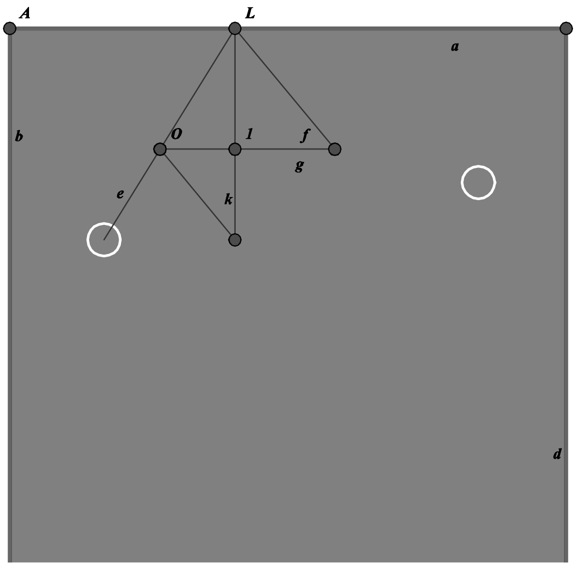

Figure 9

Example 3: Ball hits ball hits ball (centrically straight)

This is the classical situation in Carom Billiards (except for omitting the cushions and requiring a centrically straight hit of the third ball). Both models with and without loss of kinetic energy are interesting. In the simpler model without loss of kinetic energy a hard construction can be performed making constructive use of the theorem of Thales. This is shown in figure 10. The centre of the first ball when hitting the second one must lie on a circle around the centre of the second ball (having the ball diameter as radius) and also on the Thales circle over the line segment that connects the centres of balls two and three. So, the Theorem of Thales is used to construct a right angle. One could again start with a soft construction getting aware that one has to construct a right angle and then "remind" oneself that Thales' theorem is just about right angles.

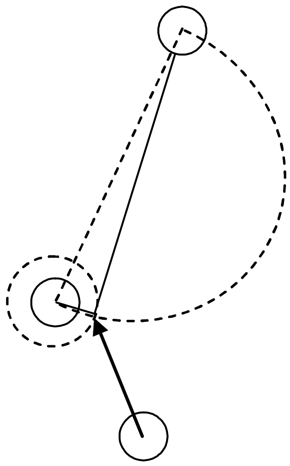

Figure 10

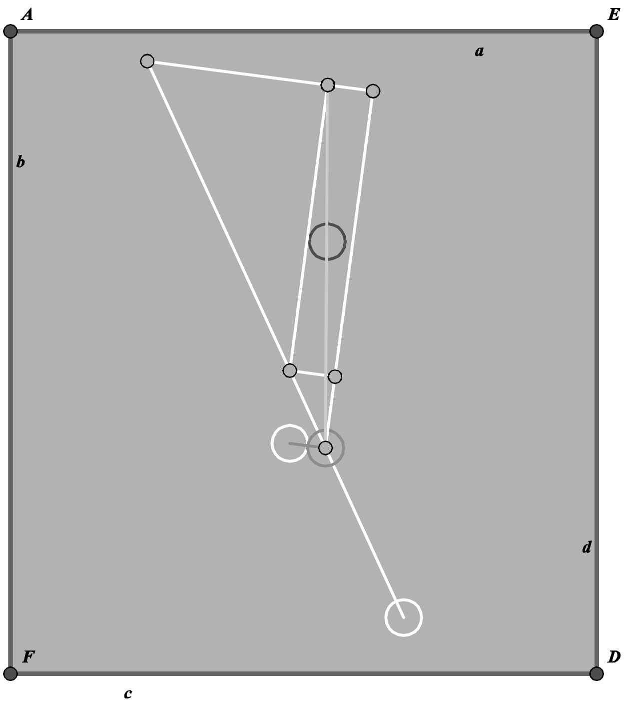

Figure 11

A larger challenge is provided by the model that includes loss of kinetic energy. Figure 11 shows a soft construction in Cinderella. It shows the first ball (below) and the second ball as white circles, the first ball when it hits the second one, and the third ball. The construction using parallel lines in the upper half of the figure guarantees that the tangens equation given in section 2 holds. When one rotates the the velocity direction of the first ball and stops when the grey line “optically” contains the centre of the third ball (as shown in figure 11), one obtains an approximate solution. Surprisingly (to the author), dragging reveals the existence of another solution that is shown in figure 12. This solution has just theoretical value since ball1 hits ball2 nearly centrically straight such that there is not enough (absolute) velocity left for the first ball to reach the third one. The existence of two theoretical solutions can also be checked by using algebraic modelling.

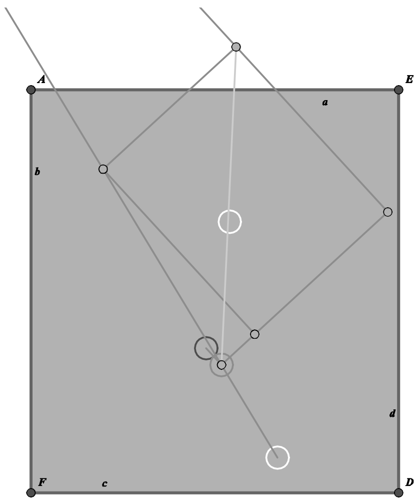

Figure 12

When trying to find a hard construction for this task using only ruler and compass which are available in any DGS, one has to fail in general. In the algebraic model one can check that the task can be transformed into the solution of an irreducible polynomial of degree 3 such that the respective points are not constructible (like the construction of a cube with a given volume). This is well above the level of understanding of students at school but might initiate at least a first treatment of the question of what kind of numbers are constructible using ruler and compass.

Soft constructions can also be used as blue prints for building mechanical devices (cf. Corves 2004 for usage of DGS in mechanism theory). The author had students develop an aiming device from the Cinderella construction in figure 11 which could be used together with the billiards machine. It should give the human player the opportunity to get an aiming direction by rotating a linkage. Figures 13 and 14 depict a “macro” and a “micro” solution for this task. The dragging mode in the DGS shows directly where there should be rotational or translational joints.

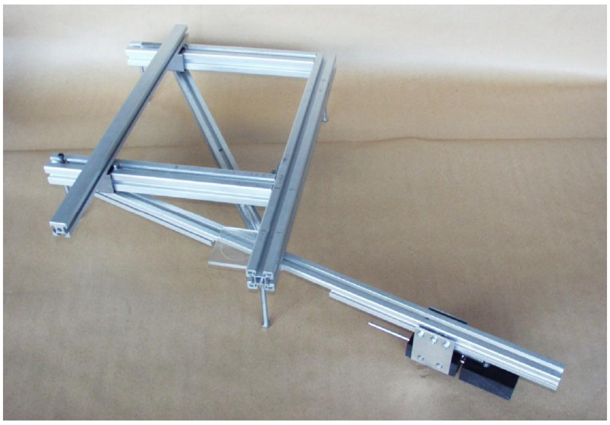

Figure 13

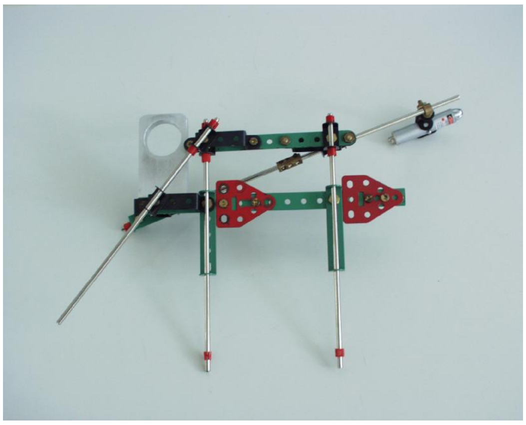

Figure 14

Example 4: Tolerance angles

The last example gives an impression of the wide variety of additional interesting tasks in the billiards scenario. In the previous task there was a restriction that a normal Carom billiard player does not have: The third ball need not be hit centrically straight, it is sufficient for a valid carambolage that the first ball just touches the third one. One can now investigate the tolerance angle (between the bold arrows in figure 15), or the sector of directions the actual velocity direction must lie in such that a carambolage occurs. How large this angle is could be considered as a measure for the difficulty of the situation. A billiards player should aim at achieving follow-up situations with a low degree of difficulty. Figure 15 shows a construction for such a tolerance angle using the usual construction methods available in a DGS. A further consideration could include the area of possible positions of the first ball such that a “ball hits ball hits ball” (without cushions) is feasible at all.

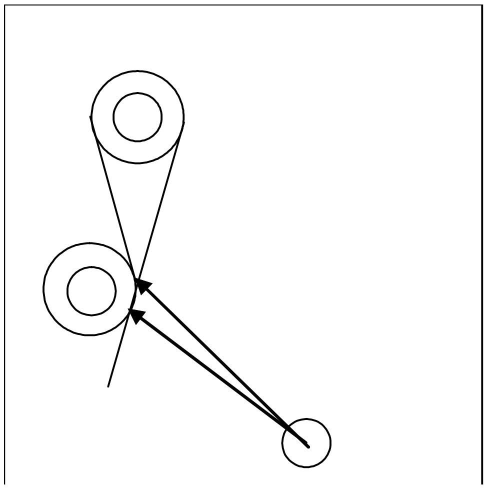

Figure 15

5. Conclusions and further work

In this contribution we have shown that the more realistic billiards model where balls are circles and friction and loss of kinetic energy can be included gives rise to a variety of interesting mathematical questions and tasks. It is a suitable topic to let students find further tasks and variations on their own (cf. Schupp 2002).
When working on these tasks, a DGS can be used in order to generate flexible constructions that can be adapted easily. Moreover, the dragging mode of a DGS can be used for performing constructions of the first class of tasks where a situation is given and the resulting run of the balls is asked for. It can also be used to create soft constructions for the second class of tasks, i.e. constructions where one condition is violated and dragging allows to find an approximate solution. Such soft constructions might give ideas for finding hard constructions and in some cases they might be the only way to find (approximate) solutions when there cannot be a hard construction due to constructability reasons. Moreover, soft constructions can also be interpreted as blue prints for aiming mechanisms.
There are also limitations when using a DGS for billiards tasks. Obviously, what is modelled are just potential motion paths. Friction is not included, and there might be other balls getting into the way. The constructions in a DGS also do not give any feedback on the correctness of the construction. Moreover, the DGS allows to easily implement geometric thoughts but it does not provide these thoughts. As was shown in the second example, the DGS might facilitate to see that one can use a certain geometric theorem or construction method (like Thales or intercepts theorems) but there must be a previous understanding of these topics that enables the learner to use them.
Concerning the feedback on real motions there are two approaches to improve the situation:

- The author implemented a Billiards microworld in the CAS Maple ® (cf. (Alpers 2002) for using CAS to implement microworlds). There, the simulated motion (in different models with and without friction) is shown in an animation (see figure 16) Moreover, having applied a velocity vector to a ball, one obtains information on all the events (collisions, rest positions) and the position and velocity vectors of all balls (see figure 17). This can be used in additional tasks when a certain configuration after a carambolage is to be achieved.

- In a beta version of Cinderella 2, there is an enhancement called CindyLab® where masses, forces and kinematic aspects can be included but in the current state, this is not sufficient for our billiards situation.

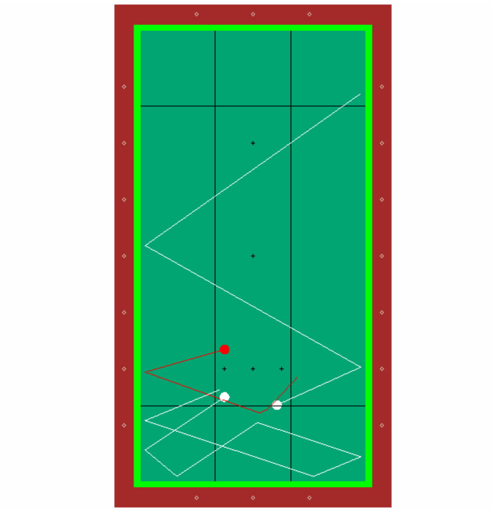

Figure 16

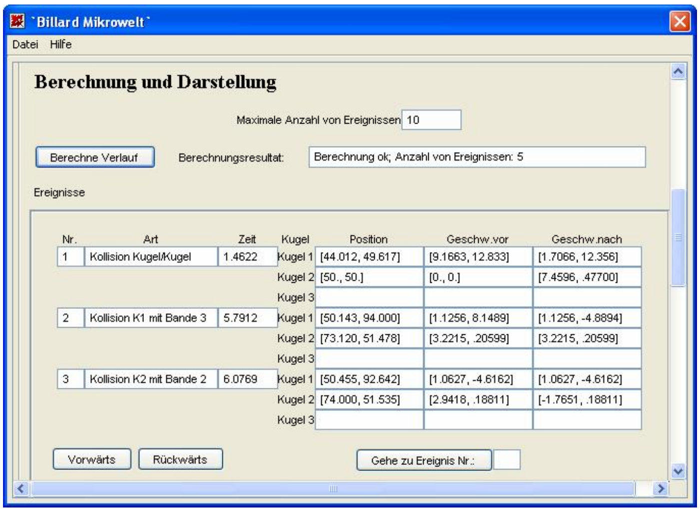

Figure 17

Having a microworld and a DGS with both interfaces to learn seems to create a “technological overkill”. Therefore, the billiards situation provides a good example for a topic which would profit from having the construction facilities of a DGS and the computational facilities of a CAS in one tool. It still has to be checked whether the available prototypes like Geogebra ® or Felix® are adequate.

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