S1 Mathematical and Physical Pendulum

Subject Area: Oscillations in general, mathematical pendulum, physical pendulum, Steiner´s theorem

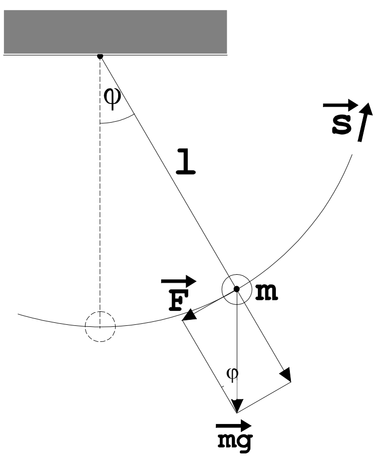
Objective of the Experiment:
Mathematical analysis of oscillatory processes (approximations, general solution methods) and measurement of gravitational acceleration g.

Literature: Textbooks "Physics for Engineers"

1. Fundamentals

1.1 Mathematical Pendulum

A mathematical pendulum can be approximated by a ball suspended from a string, where the string's mass is negligible, and the ball's mass is assumed to be concentrated at its center of gravity.



When displaced from its equilibrium position, the pendulum describes a circular arc with a radius **l**. Upon displacement by an angle φ a restoring force acts:

$$\mathbf{F = m \cdot g \cdot \sin\varphi}$$

Using $\mathbf{s = l\,\varphi}$, Newton's second law gives the equation of motion(acceleration **a** has the opposite direction to the deflection **s**):

$$\mathbf{F = m \cdot a = -\,m\frac{d^2s}{dt^2} = -\,m \cdot l\frac{d^2\varphi}{dt^2} = m \cdot g \cdot \sin\varphi}$$

$$\mathbf{\frac{d^2\varphi}{dt^2} = -\frac{g}{l}\sin \cdot \varphi}$$

l = Length of the pendulum (distance from the suspension point to the ball's center).

The differential equation for this non-harmonic oscillation cannot be solved exactly. For small angular displacements, the approximation $\mathbf{\sin\varphi \approx \varphi}$ simplifies the equation:

$$\mathbf{\frac{d^2\varphi}{dt^2} = -\frac{g}{l}\varphi}$$

The solution to this differential equation is:

$$\varphi = \varphi_0 \sin(\omega \cdot t + k)$$

Here, **k** is an integration constant, φ_0 is the amplitude, and the angular frequency **ω** is given by:

$$\omega = \sqrt{\frac{g}{l}}$$

From this, the approximate oscillation period of the mathematical pendulum for small displacements is obtained:

$$T = \frac{2 \cdot \pi}{\omega} = 2 \cdot \pi \cdot \sqrt{\frac{l}{g}} \qquad (1a)$$

Without restricting to small pendulum amplitudes, the oscillation period can be represented using a series expansion:

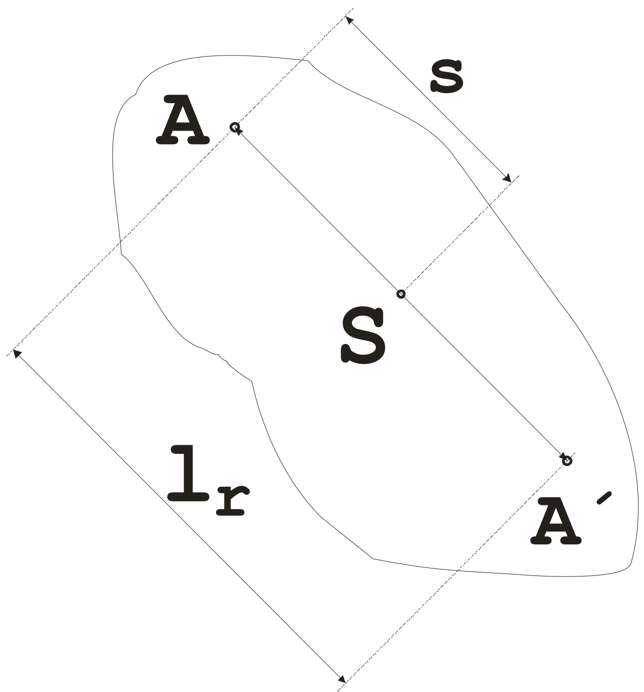
$$T = 2 \cdot \pi \sqrt{\frac{l}{g}} \left[1 + \left(\frac{1}{2}\right)^2 \cdot \sin^2\frac{\varphi_0}{2} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \cdot \sin^4\frac{\varphi_0}{2} + \ldots\right] \qquad (1b)$$

1.2 Physical Pendulum

A physical pendulum refers to any rigid body oscillating under the influence of its weight. The period of a physical pendulum is given by:

$$T = 2 \cdot \pi \sqrt{\frac{J_A}{m \cdot g \cdot s}} \qquad (2)$$

(Derivation of this formula: see the experiment with Maxwell's wheel)



A = suspension point

S = center of gravity

l_r = reduced pendulum length

s = distance from the suspension point to the center of gravity

J_A= moment of inertia of the body with respect to the axis passing through the suspension point

m = total mass of the physical pendulum

With the help of Steiner's theorem, Equation (2) can be written in the following form:

$$T = 2 \cdot \pi \sqrt{\frac{J_S + m \cdot s^2}{m \cdot g \cdot s}} \qquad (3)$$

J_S = Moment of inertia of the body with respect to the axis passing through the center of gravity, with the axis direction parallel to the axis in Equation (2).

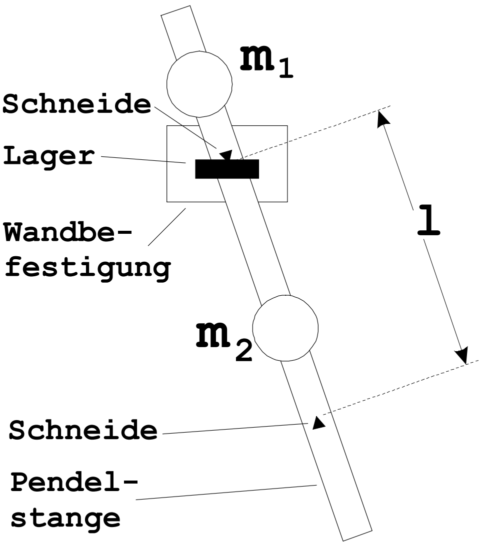
The reduced pendulum length lr is the length of the mathematical pendulum that has the same oscillation period as the physical pendulum:

$$l_r = \frac{J_S + m \cdot s^2}{m \cdot s}$$

If the pendulum is now suspended at the oscillation center **A´** (which is the point on the line connecting the suspension point **A** and the center of gravity **S**, at a distance $\mathbf{l_r}$ from **A**), the same reduced pendulum length $\mathbf{l_r} = \mathbf{l_r}´$ is obtained. Therefore, the oscillation period of a physical pendulum does not change if the oscillation center is made the pivot point.

1.3 Reverse pendulum

The reverse pendulum is used for the precise determination of the gravitational acceleration g. The property is exploited that the oscillation period of a physical pendulum does not change if the oscillation center is made the pivot point.



The reverse pendulum consists of a metal rod with two movable metal discs and two opposing edges as suspension devices. The distance between the two edges is engraved on the back of the pendulum. One metal disc with a mass of $\mathbf{m_2}$ = 1400g can be moved between the two edges, and the other disc with a mass of $\mathbf{m_1}$ = 1000g can be moved outside of one edge.

To make the pendulum oscillate, a wall mount is used, which has a bearing into which the pendulum with the edge is suspended. The pendulum is made to swing alternately around one of the two edges, and by shifting the two masses, the oscillation period of the pendulum around both edges can be made equal. Then, the length **l** is equal to the length of a similarly oscillating mathematical pendulum, and from Equation (1), the gravitational acceleration can be calculated.

2. Experimental Procedure

2.1 Measurement Method: Mathematical Pendulum

2.1.1 Measure the oscillation period of a mathematical pendulum from 50 oscillations (3 times).

2.1.2 Calculate the gravitational acceleration from the oscillation period of the mathematical pendulum.

2.1.3 Calculate the relative error of the gravitational acceleration from the errors of the individual measurement quantities and provide an estimate for the size of other errors that arise from the approximation methods used in the derivation of Equations (1) and (2).

2.2 Measurement Method: Physical Pendulum

2.2.1 By measuring the oscillation periods around both edges of the revers pendulum and shifting the inner mass, the position of the mass is determined where the oscillation period around both edges is the same. Practically, it is best to proceed by shifting the inner mass located between the edges.

The easiest way to find the position of the sliding mass, where the pendulum oscillates with the same period around both axes, is to shift the sliding mass from one edge to the other in 10 cm intervals and measure the oscillation period around both axes. For the coarse measurement, 10 oscillations are measured. The position of the sliding mass is plotted on the x-axis, and the period duration is plotted on the y-axis in a coordinate system.

By connecting the points corresponding to the same pivot axis, two curves are obtained, which must intersect at two points.

Once the intersection points are approximately determined, a fine measurement is carried out near the more suitable intersection point (justification!). The sliding mass is shifted in 1 cm intervals from 3 cm before the approximate intersection point to 3 cm after it, and the period duration around both axes is measured again. For the fine measurement, 20 oscillations are measured. The measurement points are plotted as in the coarse measurement, and the sought intersection point is determined. The scale for both the coarse and fine measurements should be chosen so that the resulting curves stretch across an A4 sheet of paper. At least the fine measurement should be recorded on graph paper.

2.2.2 Calculate the gravitational acceleration from the oscillation period of the revers pendulum.

2.2.3 Calculate the relative error of the gravitational acceleration from the errors of the individual measurement quantities and provide an estimate for the size of other errors that arise from the approximation methods used in the derivation of Equations (1) and (2).

2.3.1 Calculate the gravitational acceleration for Aalen as accurately as possible. The following formulas should be used for this:

$\mathbf{g_0 = 9{,}78049\ \frac{m}{s^2}}$ = Gravitational acceleration at the equator at sea level

$\mathbf{g_{h=0}}$ **=** Gravitational acceleration at a location with geographical latitudeφ at sea level

$$\mathbf{g_{h=0} = g_0\,(1 + 0{,}0052884 \cdot \sin^2\varphi\ -\ 0{,}0000059 \cdot \sin^2(2\,\varphi))}$$

$\mathbf{g_h}$ **=** Gravitational acceleration at a location with geographical latitude φ at height h

(air correction)

$$\mathbf{g_h = g_{h=0} - c_1 \cdot h \quad mit \quad c_1 = 3{,}086 \cdot 10^{-6}\ \frac{1}{s^2}}$$

$\mathbf{g_h}$ increases additionally by $\Delta\mathbf{g}$ due to a rock slab with density ρ and height **h**.

$$\mathbf{\Delta g = c_2 \cdot \rho \cdot h} \ \text{mit}\ \mathbf{c_2 = 4{,}19 \cdot 10^{-10}\ \frac{m^3}{kg \cdot s^2}}$$

Average density of the Earth at the Earth's surface:ρ = 2,5 g/cm^3

Aalen: 48.83° North latitude, altitude: 477.1 m

2.3.2 Compare the values from 2.1.2 and 2.2.2 with the value from 2.3.1.
What are the percentage deviations?
Compare the results with the relative errors.

3. Questions about the experiment and the subject area.

3.1 What approximations are made in the derivation of the formulas for the oscillation periods of the mathematical and physical pendulum?

3.2 What factors does the gravitational acceleration at a specific location depend on? What is the reason for this?

3.3 What terms are used in physics to describe an oscillation?

3.4 What is generally understood by an oscillation, and what specifically is meant by a harmonic oscillation?

3.5 How are the equations (2) and (3) derived?

3.6 What does Steiner's theorem state?

3.7 Calculate the error in the oscillation period of a mathematical pendulum using the approximation formula (1a) instead of formula (1b) for amplitudes $\varphi_0 = \mathbf{5^0}$ and $\varphi_0 = \mathbf{10^0}$