M1 Maxwell's Wheel

Subject area: Translational and rotational motion, moment of inertia, physical pendulum

Experiment goal: To determine the moment of inertia of a Maxwell's wheel in two ways.

Literature: Lecture manuscript, standard textbooks "physics for engineers"

1. Fundamentals

1.1 Maxwell's Wheel

The Maxwell's wheel shown in Fig. 1 is suspended with its horizontal axis of radius r on two vertical threads, so that they wind or unwind as the wheel rotates around the axis. If the wheel is brought to the highest position by winding the threads and then released, it moves downward under the influence of gravity with constant acceleration. Its motion consists simply of a combination of translational and rotational motion around its center of mass. By measuring the translational acceleration, the moment of inertia (in short, the inertia) with respect to the axis of rotation can be calculated.

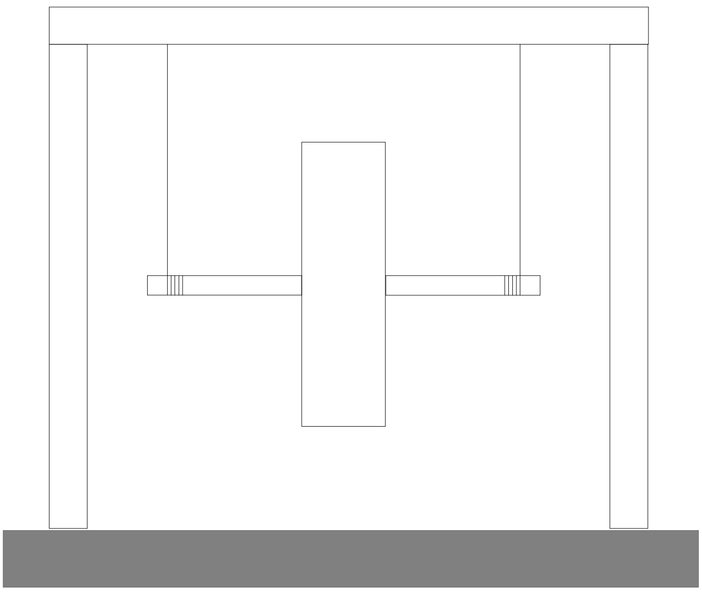

Fig.1:
Maxwell´s wheel rolls off two threads

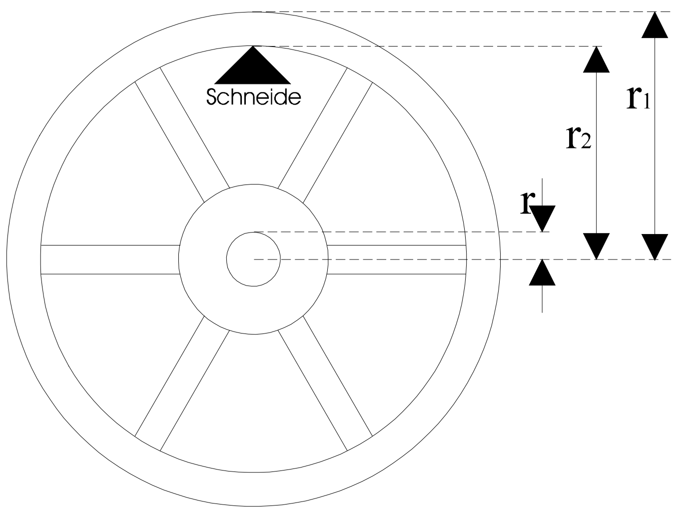

Fig.2:
Maxwell´s wheel as physical pendulum

Another method to determine the moment of inertia of the Maxwell wheel is to let it swing around an axis as a physical pendulum and, from the measured oscillation period, first calculate the moment of inertia with respect to the axis of rotation (see Fig. 2).
The fundamentals necessary to solve the tasks should be sufficiently familiar from the lecture, but they are briefly summarized again below.

1.2 Translational and Rotational Motion

1.2.1 Translational Motion

A force **($\vec{F}$)** causes a freely moving body (with mass m) to experience an acceleration ($\vec{a}$). The relationship is given by Newton's second law:

$$\vec{F} = m \cdot \vec{a} \tag{1}$$

f the force and thus the acceleration are constant, the following equations apply between acceleration, velocity $\vec{v}$, displacement $\vec{s}$, and time **t**:

$$\vec{v} = \vec{a} \cdot t + \vec{v}_0 \tag{2}$$

$$\vec{s} = \frac{\vec{a}}{2} \cdot t^2 + \vec{v}_0 \cdot t + \vec{s}_0 \tag{3}$$

Here, the velocity $\vec{v}_0$ and displacement $\vec{s}_0$ are at time t=0.
If the body has velocity v, its kinetic energy (specifically translational energy) is:

$$E_{tr} = \frac{m}{2} \cdot v^2 \tag{4}$$

1.2.2 Rotational motion

A body with mass mmm rotates around a fixed axis with angular velocity ω. To determine its kinetic energy (specifically rotational energy) E_{rot}, it must be considered that the mass elements have different velocities. The summation (integration) over the translational energy of sufficiently small mass elements gives the rotational energy of the body:

$$E_{rot} = \frac{\omega^2}{2} \bullet \int_K r^2 \bullet dm$$

(5)

The integral extended over all mass elements dm of the body.

$$J = \int_K r^2 \bullet dm \tag{6}$$

is called the moment of inertia of the body with respect to the given axis of rotation (r = distance of the mass elements from the axis of rotation).
For example, from equation (6), the moment of inertia of a homogeneous circular disk with radius r and mass m with respect to its axis of symmetry is calculated as:

$$J_S = \frac{1}{2} \bullet m \bullet r^2 \tag{7}$$

Steiner's theorem:

If m is the mass of a body, and b is the distance of an axis from the parallel axis through the center of mass (with moment of inertia Js), then the moment of inertia of the body with respect to that axis is:

$$J = J_S + m \bullet b^2 \tag{8}$$

(Steiner's theorem; see manuscript or textbook).
Thus, if the moment of inertia of a body with respect to its center of mass axis is known, its moment of inertia with respect to any other axis parallel to this center of mass axis can be calculated using this equation.

Dynamic fundamental law of rotation, laws of motion

If a torque acts on a body

$$\vec{M} = \vec{r} \times \vec{F} \tag{9}$$

(Vector product of force $\vec{F}$ and radius vector $\vec{r}$, the vector from the pivot point to the point of application of the force), the body experiences an angular acceleration α. The relationship is given by the dynamic fundamental law of rotation:

$$\vec{M} = J \cdot \vec{\alpha} \tag{10}$$

If the torque is constant, the following equations, analogous to equations (2) and (3), apply between angular acceleration α, angular velocity ω, angular displacement φ, and time t:

$$\omega = \alpha \cdot t + \omega_0 \tag{11}$$

$$\varphi = \frac{\alpha}{2} \cdot t^2 + \omega_0 \cdot t + \varphi_0 \tag{12}$$

Here, ω_0 is the angular velocity and φ_0 is the angular displacement at time t=0.

1.2.3 Application to Maxwell´s wheel

The general motion of a body consists of both translational motion and rotational motion:

Translational motion:

The center of mass of the body undergoes translation, with its acceleration $\vec{a}$ determined by equation (1), as if the resultant force were applied at the center of mass and the entire mass were concentrated at that point.

Rotational motion:

Additionally, the body undergoes rotational motion around its center of mass, with the angular acceleration determined by equation (10).

The resulting motion is generally complex, for example, with free axes. In our problem, the Maxwell wheel, it is simplified by the fact that the rotation occurs around a fixed axis, and there is a constraint between the velocity v of the center of mass and the angular velocity ω given by:

$$\mathbf{v = r \cdot \omega} \tag{13}$$

As a starting point for the calculation, the work-energy principle of mechanics can be used. If the Maxwell wheel falls from rest from a height h, this results in:

$$\frac{J}{2} \cdot \omega^2 + \frac{m}{2} \cdot v^2 = m \cdot g \cdot h \tag{14}$$

with: **v** velocity,

ω Angular velocity after traveling the height h

g Gravitational acceleration

From equations (2) and (3), it follows:

$$J = m \cdot r^2 \cdot \left(\frac{g}{a} - 1\right) \quad \text{mit } a = \frac{2 \cdot h}{t^2} \tag{15}$$

The same result can be obtained using equations (1) and (10). Two forces act on the Maxwell wheel: the gravitational force and the tension in the rope (see Fig. 3).

This leads to: $m \cdot a = m \cdot g - F_{Seil}$

Motion of the center of mass:

$$m \cdot a = m \cdot g - F_{Seil} \tag{16}$$

Rotation around the center of mass axis:

$$J \cdot \alpha = r \cdot F_{Seil} \tag{17}$$

The elimination of $\mathbf{F_{Seil}}$ from these two relationships, along with equation (13), immediately leads to the result in equation (15) (after which a=rα).

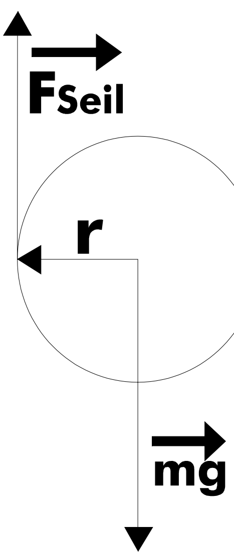

Fig. 3

1.3 Physical pendulum

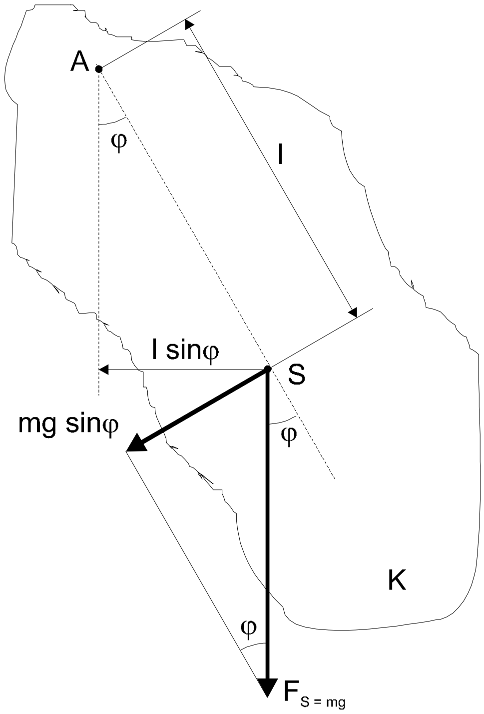

Fig. 4

A body is rotatable about an axis AAA (not the center of mass axis). The distance from A to the center of mass S is l. When the body is displaced by an angle φ from its resting position a restoring torque acts:

$$\mathbf{M = -l \bullet m \bullet g \bullet \sin\varphi} \qquad (18)$$

(See Fig. 4.). According to equation (10), the equation of motion then results in:

$$\mathbf{-l \bullet m \bullet g \bullet \sin\varphi = J_A \bullet \frac{d^2\varphi}{dt^2}}$$

and, respectively,

$$\mathbf{\frac{d^2\varphi}{dt^2} + \frac{l \bullet m \bullet g}{J_A} \bullet \sin\varphi = 0} \qquad (19)$$

The solution of this nonlinear differential equation results in anharmonic oscillations. For sufficiently small displacements φ, it can be assumed that:

$$\mathbf{\sin\varphi \approx \varphi} \qquad (20)$$

The solution of the linearized differential equation is:

$$\mathbf{\varphi = \varphi_0 \bullet \sin(\omega t + \beta_0)} \qquad (21)$$

with

$$\omega = \sqrt{\frac{l \cdot m \cdot g}{J_A}} \qquad \text{bzw.} \quad T = 2 \cdot \pi \cdot \sqrt{\frac{J_A}{l \cdot m \cdot g}} \qquad (22)$$

Here, T is the period of oscillation, and φ_0 and β_0 are integration constants. If J is the moment of inertia about the center of mass axis parallel to the axis of rotation, then Steiner's Theorem (equation (8)) gives:

$$J = J_A - m \bullet l^2 \tag{23}$$

From equation (22) and equation (23),

$$J = \frac{T^2 \bullet l \bullet m \bullet g}{4 \bullet \pi^2} - m \bullet l^2 \tag{24}$$

By measuring the period of oscillation, the sought moment of inertia J can thus be determined.

2. Experimental procedure:

2.1 Maxwell´s wheel rolls off two threads.

2.1.1 Suspend Maxwell´s wheel on two equal-length, thin threads that converge slightly toward the suspension mechanism, with a horizontal axis. Rotate the wheel (the threads wind onto the axis) until it is raised to height h.

2.1.2 Measurement: The wheel is released, and the time t is measured for it to pass through the height h. At least 10 individual measurements should be taken.

2.1.3 The moment of inertia can be calculated using equation (15).

2.1.4 To what extent does the thickness of the thread play a role?
Correct the result.

2.1.5 Assuming that m, r, t, and h are subject to errors, the maximum relative error is given by:

$$\frac{\Delta J}{J} = \frac{\Delta m}{m} + 2 \bullet \frac{\Delta r}{r} + \frac{1}{1 - \frac{2 \bullet h}{g \bullet t^2}} \bullet \left(\frac{\Delta h}{h} + 2 \bullet \frac{\Delta t}{t}\right) \tag{25}$$

Calculate this and discuss the result.

2.2 Wheel swinging over edge

2.2.1 Let the wheel swing over the edge. Determine the oscillation period from 50 oscillations. Repeat the experiment twice.
2.2.2 Calculation of the moment of inertia using equation (24).
(According to Figure 2, **l = r2**
2.2.3 Assuming that T, l, and m are subject to errors, the maximum relative error is obtained as:

$$\frac{\Delta J}{J} = \frac{\Delta m}{m} + \left|\frac{1 - 2 \bullet u}{1 - u}\right| \bullet \left|\frac{\Delta l}{l}\right| + \left|\frac{2}{1 - u}\right| \bullet \left|\frac{\Delta T}{T}\right| \quad \text{mit} \quad u = \frac{4 \bullet \pi^2 \bullet l}{g \bullet T^2} \qquad (26)$$

Calculate this and discuss the result

2.3 Rule of thumb

2.3.1 For the calculation of the moment of inertia, the rule of thumb can be applied.

$$J = k \bullet m \bullet \frac{r_1^2 + r_2^2}{2} \qquad (27)$$

For **k = 1**, the moment of inertia corresponds to that of a thick-walled hollow cylinder. By setting **k < 1**, it takes into account that part of the mass of the wheel is located in the spokes.
2.3.2 Determine the factor k for the Maxwell wheel (based on the measured moment of inertia).
2.3.3 Calculate the maximum relative error of k when m, r1 , r2 and J are error-prone.

$$\frac{\Delta k}{k} = \frac{\Delta J}{J} + \frac{\Delta m}{m} + \frac{2 \bullet r_1 \bullet \Delta r_1 + 2 \bullet r_2 \bullet \Delta r_2}{r_1^2 + r_2^2} \qquad (28)$$

2.4 Comparison of the Methods

2.4.1 Compare the maximum relative errors from 2.1.5 and 2.2.3.

2.4.2 What error sources, aside from those already considered in the error calculation, occur? List them.

2.4.3 Which of the two measurement methods is more accurate? Explain why.

3. Questions about the experiment and subject area

3.1 In translational motion, mass is a measure of the body's inertia. Does this also apply to rotational motion?

3.2 List some examples from everyday life and technology where the moment of inertia of a body is significant.

3.3 What does Steiner's theorem state? How can it be derived?

3.4 Energy theorem of mechanics: Under what conditions does it apply? What is neglected in equation (14)?

3.5 Show that equation (15) follows from equations (13) and (14).

3.6 Equation (17): Why does only the rope force create a torque and not the weight force?

3.7 Explain the terms "mathematical pendulum" and "physical pendulum."

3.8 How would you explain the sentence on page 6, "...for sufficiently small displacements, it can be assumed..." in more detail?

3.9 Derive the formulas for the maximum relative error of J from equations (25) and (26).

3.10 How is the quantity Δt determined in equation (25)?