F Guide to Error Calculation in Physics Lab

1. Definition of Error

1.1 Definition des Fehlers

a) Absolute Error

Measured values and results calculated from measured values are always subject to errors. The absolute error **Δx** is the difference between the measured actual value of the measured quantity **x** (measured value) and the true value of the measured quantity **x_w**.

$$\Delta x = x - x_w .$$

The absolute error has the dimension of the measured quantity.

b) Relative Error

Often, the relative error is also calculated:

$$\text{Relativer Fehler} = \frac{\text{absoluter Fehler}}{\text{wahrer Wert}} = \frac{\Delta x}{x_w}$$

Since the true value **x_w** of a measured quantity is never precisely known, the error (assuming the error is "sufficiently small") is related to the measured value **x**:

$$\text{Relativer Fehler} = \frac{\Delta x}{x} \qquad \left(\text{Rel. Fehler in \%} = \frac{\Delta x}{x} \cdot 100\right)$$

1.2 Systematic Errors

Systematic errors are errors that originate from the measurement system. They are reproducible and occur with the same magnitude and direction when the measurement is repeated. Systematic errors cause measurements to be incorrect.

Example: A ruler or measuring instrument has been improperly calibrated.

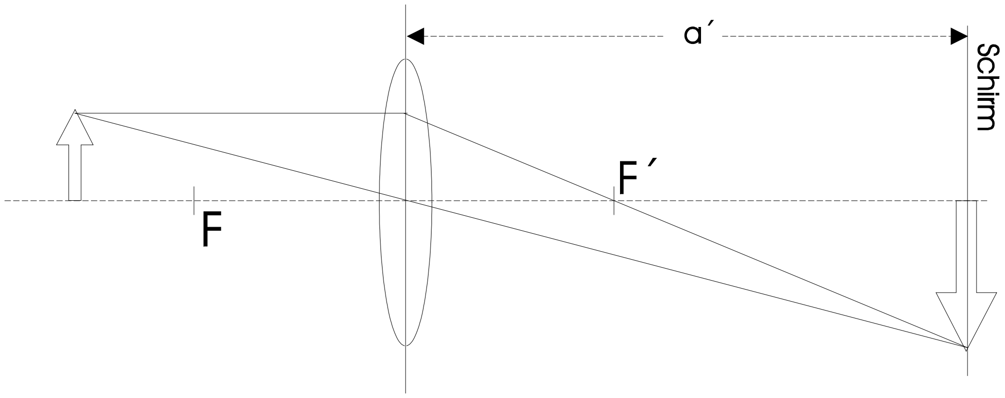
This type of systematic error can only be detected through inspection and eliminated by recalibrating the measuring devices.

Systematic errors may also arise due to observer mistakes, the use of inappropriate measurement methods, etc.

A significant cause of systematic errors is the "quality" of the measuring instruments. For example, with a simple caliper, lengths can only be measured with an accuracy of ±0.1 mm.
This type of systematic error can only be reduced by improving the measurement system (Caliper >> Micrometer screw gauge).

1.3 Random Errors

Random errors are non-reproducible, meaning they differ in magnitude and amount when repeating measurements of a constant quantity with the same measuring instrument. Random errors make a measurement inaccurate.
Example: Measurement of the image distance a’ with the following simple lens arrangement.



By shifting the screen, the image is focused. Since the sharpness cannot be judged exactly, the measured values of a´ scatter. The deviations from the true image distance $a_W´$ are generally different in magnitude and direction, as the image is randomly recognized as sharp either earlier or later.
Random errors can be reduced by increasing the number of measurements. They are theoretically quantifiable (Gaussian error theory).
Random errors and systematic errors cannot always be strictly separated. The total error usually results from the combination of both types of errors.

2. Computational detection of randomly distributed errors

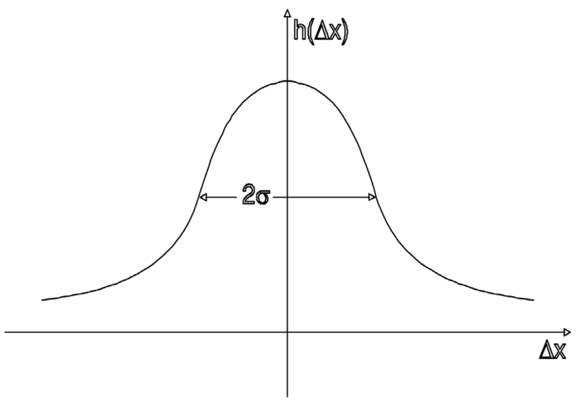
The representation of the relative frequency of errors h(Δx) as a function of the absolute error Δx is called the error distribution.

n practice, it is common and also reasonable based on long experience to assume a Gaussian distribution (normal distribution) as the distribution form of random errors:

$$h(\Delta x) = \frac{1}{\sigma \bullet \sqrt{2\pi}} \bullet e^{-\frac{1}{2}(\frac{\Delta x}{\sigma})^2}$$

2.1 Mean

For measurements affected by random errors, multiple measurements can be taken to increase measurement accuracy $n \geq 10$. The best value is the arithmetic mean:



$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

2.2 Mean error of the individual value σ and mean error of the mean value Δx

The mean error of the individual value σ (standard deviation) is a measure of the deviation of the individual measurement value from the mean value $\bar{x}$ (width of the Gaussian bell curve).

$$\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{\sum \left(x_i - \frac{\sum x_i}{n}\right)^2}{n-1}}$$

n	number of measurements
$\bar{x}$	mean value
x_i	Individual measurement values

(σ is also referred to as the sample standard deviation.)

The characteristic σ indicates the following:

If any x is measured, it will lie within the limits of

$\bar{x} \pm \sigma$	68,3%
$\bar{x} \pm 2\sigma$	95,4%
$\bar{x} \pm 3\sigma$	99,7%

for all measurement values.

The mean error of the mean value Δx is smaller than the mean error of the individual value σ:

$$\Delta x = \frac{t \cdot \sigma}{\sqrt{n}} = t \cdot \sqrt{\frac{\sum (x_i - \bar{x})^2}{n \cdot (n-1)}}$$

For t = 1,2,3 the probability of a repeatedly measured mean $\bar{x}$ falling in the interval $\bar{x} \pm \Delta x$ is 68,3%, 95,4%, 99,7%, respectively.

In physics and metrology, a statistical confidence of 68.3% is typically considered sufficient, so t = 1 is used:

$$\Delta x = \frac{\sigma}{\sqrt{n}}$$

(In industry, t = 2 is preferred, while in biology, t = 3 is preferred.)

2.3 Example for calculating the mean, standard deviation, and standard error of the mean.

In the example mentioned under 1.3 for measuring the image distance a', 10 values of a' were measured. The following calculation scheme results:

$a_i' \left[m\right]$	$(a_i' - \bar{a}')\left[m\right]$	$(a_i' - \bar{a}')^2 \left[m^2\right]$
0,600	$1 \cdot 10^{-3}$	$1 \cdot 10^{-6}$
0,601	0	0
0,598	$3 \cdot 10^{-3}$	$9 \cdot 10^{-6}$
0,605	$-4 \cdot 10^{-3}$	$16 \cdot 10^{-6}$
0,603	$-2 \cdot 10^{-3}$	$4 \cdot 10^{-6}$
0,600	$1 \cdot 10^{-3}$	$1 \cdot 10^{-6}$
0,603	$-2 \cdot 10^{-3}$	$4 \cdot 10^{-6}$
0,600	$1 \cdot 10^{-3}$	$1 \cdot 10^{-6}$
0,601	0	0
0,599	$2 \cdot 10^{-3}$	$4 \cdot 10^{-6}$

$$0{,}601 \ \mathrm{m} = \bar{a}'$$

$$40 \bullet 10^{-6} \ \mathrm{m}^2 = \sum_{i=1}^{10} (a_i' - \bar{a}')^2$$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{10} (a_i' - \bar{a}')^2}{n-1}}$$

$$\sigma = \sqrt{\frac{40 \bullet 10^{-6}}{9}} = 0{,}00210815 \ \mathrm{m}$$

$$\Delta a' = \frac{\sigma}{\sqrt{n}}$$

$$\Delta a' = \frac{\sigma}{\sqrt{10}} = 0{,}000666 \ \mathrm{m}$$

Final result: <u>a´ = (0,601 ± 0,000666) m</u>

3. Error propagation

Frequently, a measurement result y is derived from several measured values x_i, which are related by a functional relationship:

$$\mathbf{y = f(x_1, x_2, x_3, ...)}$$

The measured values xi are subject to systematic or random errors (Δx_i). Since the accuracy of the measurement result y often has specific requirements, answering the question of how the errors Δx_i affect y ("error propagation") is crucial. Only then can it be determined which quantities need to be measured with particular care and which can be measured with less precision. Both aspects influence the effort required for the measurement.

3.1 Fundamentals of error propagation:

Since, in general, the error Δx_i of a measured quantity $\mathbf{x_i}$ is not precisely known, but only the error limits **±** $\mathbf{\Delta x_i}$, it is often practical, in error propagation, to assume the worst-case scenario. That is, the case where all occurring errors "add up" rather than canceling each other out.

The mathematical foundation of error propagation (assuming $\mathbf{\Delta x_i << x_i}$) is the so-called total differential:

$$\Delta \mathbf{y} = \left| \frac{\partial \mathbf{y}}{\partial \mathbf{x_1}} \bullet \Delta \mathbf{x_1} \right| + \left| \frac{\partial \mathbf{y}}{\partial \mathbf{x_2}} \bullet \Delta \mathbf{x_2} \right| + ...$$

(Absolute values due to maximum estimation)

If y depends on only one measured value x (y = f(x)), the "differential" is given by:

$$\Delta \mathbf{y} = \left| \mathbf{f'(x)} \bullet \Delta \mathbf{x} \right|$$

However, it is not always necessary to calculate using the general formulas; often, applying pre-derived formulas for commonly occurring terms is sufficient (see section 3.2).

3.2 Special calculation rules for error propagation:

$$y = f(x_1, x_2) = x_1 \pm x_2 \qquad \Delta y = |\Delta x_1| + |\Delta x_2|$$

Multiplication and division of two measured quantities:

$$y = C \cdot x_1 \cdot x_2 \quad \text{bzw.} \quad y = C \cdot \frac{x_1}{x_2} \qquad \frac{\Delta y}{y} = \left|\frac{\Delta x_1}{x_1}\right| + \left|\frac{\Delta x_2}{x_2}\right|$$

Power and square root (of a measured quantity):

$$y = C \cdot x^n \qquad \frac{\Delta y}{y} = |n| \cdot \left|\frac{\Delta x}{x}\right|$$

$$y = \sqrt[n]{x} \qquad \frac{\Delta y}{y} = \left|\frac{1}{n}\right| \cdot \left|\frac{\Delta x}{x}\right|$$

Power product:

$$y = f(x_1, x_2, \ldots) = C \cdot x_1^k \cdot x_2^m \cdot x_3^n \cdot \ldots \qquad \frac{\Delta y}{y} = |k| \cdot \left|\frac{\Delta x_1}{x_1}\right| + |m| \cdot \left|\frac{\Delta x_2}{x_2}\right| + \ldots$$

hier **C** = constant

Note: In addition and subtraction, the absolute error is used, while in multiplication, division, exponentiation, and rooting, the relative error is applied. The following example will demonstrate how, using these simple rules, more complex functions can also be handled.

3.3 Example

The following relationship between the quantities **m**, **r**, and **a** is given:

$$J(m,r,a) = m \cdot r^2 \cdot \left(\frac{g}{a} - 1\right)$$

Mass **m** and radius **r** are measured directly. The acceleration a is calculated from other measured quantities and, like **m** and **r**, is subject to an error. The gravitational acceleration **g = 9,81 m/s²** is considered to be error-free. What is the error ΔJ for the calculated moment of inertia J?

$$m = (0{,}515 \pm 0{,}005)\ \text{kg}; \qquad \frac{\Delta m}{m} \approx 0{,}01 = 1\%$$

$$r = (0{,}0028 \pm 0{,}0001)\ \text{m}; \qquad \frac{\Delta r}{r} \approx 0{,}04 = 4\%$$

$$a = (0{,}121 \pm 0{,}005)\ \frac{m}{s^2}; \qquad \frac{\Delta a}{a} \approx 0{,}04 = 4\%$$

Substituting the numerical values (without errors) into the formula gives:

$$J = 0{,}515\ \text{kg} \cdot 0{,}0028^2\ \text{m}^2 \cdot \left(\frac{9{,}81\,\frac{m}{s^2}}{0{,}121\,\frac{m}{s^2}} - 1\right) = 3{,}23 \cdot 10^{-4}\ \text{kg m}^2$$

a) Error calculation with special formulas:

$$J(m,r,a) = u(m,r,a) + v(r,m)$$

$$\text{wobei } u(m,r,a) = m \cdot r^2 \cdot \frac{g}{a}$$

$$\text{und } v(m,r) = -m \cdot r^2$$

u and v can be treated according to the rules for a power product:

$$\frac{\Delta u}{u} = \frac{\Delta m}{m} + 2 \cdot \frac{\Delta r}{r} + \frac{\Delta a}{a} \quad \text{und} \quad \frac{\Delta v}{v} = \frac{\Delta m}{m} + 2 \cdot \frac{\Delta r}{r}$$

For further processing, the sum rule is used for **J (u,v) = u + v**. Before that, the absolute errors are calculated from the relative errors $\frac{\Delta u}{u}$ und $\frac{\Delta v}{v}$, as the sum rule requires the use of absolute errors.

$$\frac{\Delta u}{u} \bullet u = \Delta u; \quad \frac{\Delta v}{v} \bullet v = \Delta v$$

$$\Delta J = \Delta u + \Delta v = u \bullet \left(\frac{\Delta m}{m} + 2 \bullet \frac{\Delta r}{r} + \frac{\Delta a}{a}\right) + v \bullet \left(\frac{\Delta m}{m} + 2 \bullet \frac{\Delta r}{r}\right)$$

A simplification often occurs when the relative error is calculated before substituting the numerical values.

$$\frac{\Delta J}{J} = \frac{u}{u+v} \bullet \left(\frac{\Delta m}{m} + 2 \bullet \frac{\Delta r}{r} + \frac{\Delta a}{a}\right) + \frac{v}{u+v} \bullet \left(\frac{\Delta m}{m} + 2 \bullet \frac{\Delta r}{r}\right)$$

$$= \frac{\Delta m}{m} + 2 \bullet \frac{\Delta r}{r} + \frac{g}{g-a} \bullet \frac{\Delta a}{a}$$

$$\approx 0{,}01 + 0{,}08 + 0{,}04 = 0{,}13 = \underline{13\%}$$

Final result: $\underline{\underline{J = (3{,}23 \pm 0{,}42) \bullet 10^{-3}\ \text{kg m}^2}}$

b) Error calculation with the total differential:

$$J(m, r, a) = m \bullet r^2 \bullet \left(\frac{g}{a} - 1\right)$$

$$\Delta J = \left|\frac{\partial J}{\partial m} \bullet \Delta m\right| + \left|\frac{\partial J}{\partial r} \bullet \Delta r\right| + \left|\frac{\partial J}{\partial a} \bullet \Delta a\right|$$

$$= r^2 \bullet \left(\frac{g}{a} - 1\right) \bullet \Delta m + 2 \bullet m \bullet r \bullet \left(\frac{g}{a} - 1\right) \bullet \Delta r + \left|\frac{-m \bullet r^2 \bullet g}{a^2}\right| \bullet \Delta a$$

For simplicity, the relative error is again calculated.

$$\frac{\Delta J}{J} = \frac{r^2 \bullet (\frac{g}{a} - 1)}{m \bullet r^2 \bullet (\frac{g}{a} - 1)} \bullet \Delta m + \frac{2 \bullet m \bullet r \bullet (\frac{g}{a} - 1)}{m \bullet r^2 \bullet (\frac{g}{a} - 1)} \bullet \Delta r + \frac{(m \bullet r^2 \bullet \frac{g}{a^2}) \bullet \Delta a}{m \bullet r^2 \bullet (\frac{g}{a} - 1)}$$

$$= \frac{\Delta m}{m} + 2 \bullet \frac{\Delta r}{r} + \frac{g}{g - a} \bullet \frac{\Delta a}{a}$$

It turns out that the largest contribution to the error, namely 8%, comes from the measurement of **r** This is where efforts should primarily be focused if a significant reduction in error is desired (caliper >> micrometer screw gauge). The error in mass determination (**Δm** = 5g) contributes only 1% to the total error.

4. Linear regression (least squares line)

There is a linear relationship between the directly measurable quantities x and y in the absence of measurement errors:

$$y = ax + b$$

The values of a and b are to be determined by measuring **n** pairs of values **(x_i,y_i)** (with random errors in **x_i,y_i**). The best values of a and b are then given by the following formulas:

$$\bar{a} = \frac{n \bullet \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \bullet \sum_{i=1}^{n} y_i}{n \bullet \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}$$

$$\bar{b} = \frac{\sum_{i=1}^{n} y_i \bullet \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \bullet \sum_{i=1}^{n} x_i y_i}{n \bullet \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}$$

The mean error of the individual measurement is obtained from the deviations

$v_i = y_i - \bar{b} - \bar{a} \cdot x_i$ as follows:

The mean errors m_a of $\bar{a}$ and m_b of $\bar{b}$ are given by:

$$m_a = m \bullet \sqrt{\frac{n}{n \bullet \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}} \quad m = \sqrt{\frac{\sum_{i=1}^{n} v_i^2}{n-2}}$$

$$m_b = m \bullet \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n \bullet \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}}$$

Such calculations are typically performed electronically today. In our laboratory practice, these steps are also carried out by a computer.