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Modelling and Problem Solving in Billiards using DGS and Billiards Machine

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Abstract

In this article we discuss several models for the billiards scenario and identify a typology of problems. For these, several examples are provided. Moreover, we discuss how technological tools like a DGS and a Billiards machine can be used in the problem solving process. We also report on using Billiards problems in mathematical application projects.

1. INTRODUCTION

In so-called "Mathematical Billiards" balls are modelled as points and the motion of such points on different table geometries is investigated (Drexler, Gander 1998). This model can be found frequently in the didactical literature. It is used to treat reflection geometry and reflection groups (Stowasser 1976, Shultz/Shiflett 1988), closed trajectories and irrational numbers (Barabash 2003), hull curves as well as chaotic behaviour (Bettinaglio/Lehmann 1998). Although mathematical billiards is a suitable starting point to get to interesting mathematical problems, it lacks authenticity since the real billiards scenario including several balls is not modelled. In order to overcome this problem, this contribution addresses the following questions:

- How can billiards be modelled more realistically (but not too complex) and which interesting learning opportunities and problems come up?
- How can technological tools like Dynamic Geometry Systems (DGS), and a real billiards machine support modelling and problem solving processes?

Modelling the real billiards ball motion is quite complex (Marlow 1995). Differentiating between sliding and rolling and inclusion of spin would exceed the capabilities of most high school students. Therefore, we suggest a sequence of models which is simpler, yet sufficiently realistic for describing the behaviour of a billiards machine we constructed in our lab. For these models we set up a typology of problems consisting of basically four types. Examples for these types will be presented where even in the simplest model interesting geometric questions come up. Although most of the problems we identified could also be tackled using pencil and paper, technological tools can be very useful for getting and testing ideas and hypotheses as well as for motivating model propagation. We investigate potentials and limitations of a DGS (Cinderella), and a real Billiards machine. Finally, we report on using the Billiards scenario and the tools in mathematical application projects for mechanical engineering students.

2. SEQUENCE OF MODELS

We use three levels of modelling:

- Mathematical billiards with balls and impacts

- Physical billiards including friction
- Physical billiards including friction and loss of (kinetic) energy on collisions

We do not consider any spin in our models. This is certainly a restriction with respect to modelling the human billiards player but not with respect to the billiards machine we describe below.

Mathematical billiards with balls and impacts and exchange of velocities:
In this model we consider balls which have a certain two-dimensional velocity. Since friction is not included in the model, the balls move with constant velocity until there is a collision with another ball or with a cushion. Such a collision or impact is called centrical if the centers of mass of the two bodies involved lie on the normal line through the point where the bodies touch each other at collision time. With billiards balls, this is always the case. Moreover, an impact is called centrically straight ("head-on collision") if the velocity vectors of the two balls lie on this normal line as shown in figure 1. Otherwise an impact is called oblique (cf. figure 2).

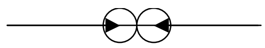

FIGURE 1. Head-on Collision

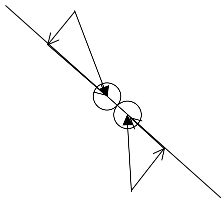

FIGURE 2. Oblique Impact

We first consider a centrically straight collision of two bodies. The principles of conservation of momentum and of energy provide equations from which one can easily compute the velocities $\overline{v}_1$ and $\overline{v}_2$ after collision (m_1 and m_2 are the masses involved, v_1 and v_2 are the respective velocities before collision)

$$\overline{v}_1 = \frac{2 \cdot m_2 v_2 + (m_1 - m_2) \cdot v_1}{m_1 + m_2}, \quad \overline{v}_2 = \frac{2 \cdot m_1 v_1 + (m_2 - m_1) \cdot v_2}{m_1 + m_2} \qquad (1)$$

Two situations are interesting here:
When a ball hits another ball, we have equality of masses and we get $\overline{v}_1 = v_2, \overline{v}_2 = v_1$. This means that the relative velocity of the two masses changes its sign but the absolute value is preserved: $\overline{v}_1 - \overline{v}_2 = v_2 - v_1$.
When a ball with mass m_1 hits a cushion, we have $m_2 \gg m_1, v_2 = 0$, so we can neglect m_1 and get $\overline{v}_1 = -v_1, \quad \overline{v}_2 = 0$ (we could also take limits for $m_2 \to \infty$).
When the collision is not centrically straight but oblique, then the velocity vectors are simply decomposed into normal and tangential components as shown in figure 2. For the normal components the considerations of the centrically straight impact can be applied whereas the tangential components are not influenced by the collision and hence do not change.

Since there is no friction in the model, the balls can move infinitely long. So, it obviously does not model physical reality and hence we still call this model "mathematical billiards". Nevertheless, working with this model can already give answers to those practical questions which are purely geometrical in nature.

Physical billiards including friction

On our next level of modelling, friction is included. We assume that the balls are always rolling (i.e. there is no sliding phase) and friction is modelled by a constant force that is proportional to the gravitational force with proportionality factor μ_r. Therefore, the distance over time function is given by:

$$s(t) = v_0 t - \frac{1}{2}\mu_r g t^2 \text{ (until } \dot{s}(t) = 0\text{)} \quad (2)$$

This model is already quite realistic but experiments with the billiards machine show that it can be quite coarse and lead to results where a desired collision is not achieved.

Physical billiards including friction and loss of kinetic energy

In this model, we assume that when two balls collide centrically straight, conservation of momentum is still given but the absolute value of the relative velocity is no longer preserved, i.e. we no longer have $\bar{v}_1 - \bar{v}_2 = v_2 - v_1$ but $\bar{v}_1 - \bar{v}_2 = k \cdot (v_2 - v_1)$ with $0 \le k \le 1$. Here, k is called the coefficient of restitution. This is equivalent to a loss of kinetic energy. Solving the equations for the velocities after collision leads to

$$\bar{v}_1 = \frac{(1+k)\cdot m_2 v_2 + (m_1 - k m_2)\cdot v_1}{m_1 + m_2}, \bar{v}_2 = \frac{(1+k)\cdot m_1 v_1 + (m_2 - k m_1)\cdot v_2}{m_1 + m_2} \quad (3)$$

There are three interesting special cases:

- If two balls collide, then the masses are equal and can be cancelled.
- If a ball collides with a cushion, then $m_2 \gg m_1$, $v_2 = 0$, and hence: $\bar{v}_1 = -k \cdot v_1$
- If the cue of the billiards machine collides with a resting ball, then $m_1 \gg m_2$, $v_2 = 0$, and hence $\bar{v}_2 = (1+k)\cdot v_1$.

The coefficients of restitution are different for collisions between balls, collisions between a ball and a cushion, and collisions between the "cue" of the billard machine and a ball. They can be derived from measurements.

3. TYPES OF TASKS IN BILLARDS

We assume that the basic underlying physical laws are already known, i.e. conservation of momentum and energy as well as motion with constant velocity or acceleration. The modelling task in Billiards then consists of applying these concepts to the Billiards scenario. For the first and second model, the modelling task is quite simple. The only challenge is to proceed from the one-dimensional to the two-dimensional situation by splitting up the velocity vector into two components. This is an important principle when considering multi-dimensional situations and hence has a value in itself. When getting to the third model, one has to introduce the coefficient of restitution which models the loss of relative velocity and hence

implicitly the loss of kinetic energy. The motivation for advancing to the third model might come from observing the motion of real balls in the Billiards machine (section 5) which shows that the law of reflection used in the second model is (more or less) violated in reality.
Once the above models have been set up, they can be used in solving a variety of problems. This problem solving activity using algebraic and geometric mathematical knowledge forms the main activity in our Billiards scenario. We identified a typology of problems consisting of basically four types where up to three balls are involved:

- Given a start configuration and an impact, what happens?
- Given a start configuration, how can a carambolage be achieved?
- Construction of configurations and impacts with given properties
- Computation of system coefficients from real data

Within each type one can start with simple problems involving just one ball and one can use the results when working on more complicated problems. There are a lot of possible variations, so that students can be asked to vary tasks or to create their own tasks. We just give an example for each of the above types.

Example 1: Given the configuration depicted in figure 3 (i.e. the positions of the balls and the velocity vector of ball1), what happens?

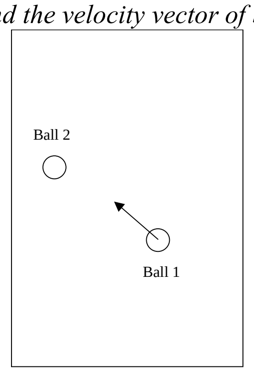

FIGURE 3. Impact computation

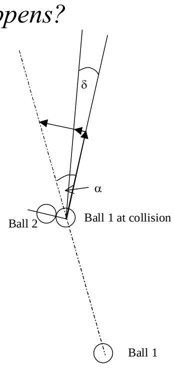

FIGURE 4. Impact with loss of energy

This task can be considered in all model types: Always, the question of whether or not ball 1 hits ball 2 comes up. This can be solved geometrically by drawing a circle with radius two times the ball radius around the center of ball 2 and intersecting it with the line through the center of ball 1 going in the direction of the velocity vector. The intersection point (and hence the position of ball 1 at collision time) can also be computed algebraically by solving the corresponding system of equations (quadratic and linear). The behaviour after collision depends on the type of model under consideration. In both "mathematical billiards" and "physical billiards including friction" the path of ball 1 after collision is orthogonal to the path of ball 2. In the physical model including loss of kinetic energy, we have a deviation from orthogonality since ball 1 still has a non-zero component in the direction of the motion of ball 2 as is shown in figure 4 (resulting in a deviation of δ where δ can be computed from $\tan\delta = \frac{1}{2}(1-k)\tan\alpha$).

<u>*Example 2:*</u> *Given the configuration depicted in figure 5 (i.e. the positions of the balls), compute a velocity vector for ball 1 such that it impacts ball 2 centrically straight?*

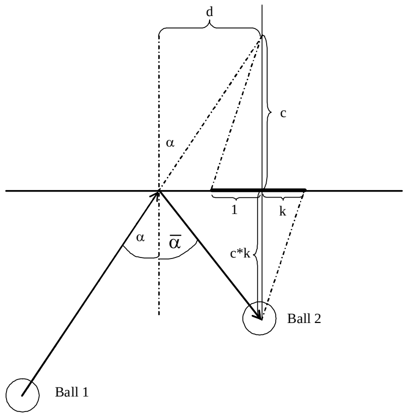

FIGURE 5. Collision with cushion and loss of energy

In the first and second model this task can simply be solved by reflecting ball 2 at the cushion (or more precisely: at a line having a distance of a ball radius from the cushion). Similar tasks can be solved for several cushions by composing reflections. In the third model, the situation is more interesting and a possible solution is depicted in figure 5. It can be solved algebraically using the equation $\tan(\alpha) = k \cdot \tan(\overline{\alpha})$ or geometrically performing a composition of reflection and dilation. This can be done by using intercept theorems as is shown in figure 5 (any multiplication can be performed geometrically this way). Connecting the image of ball 2 (i.e. its center) with the center of ball 1 provides the desired direction of the impact on ball 1.

Variations of this task include a consideration of the interval of angles such that the second ball is hit at all (the width of this interval can later be used as a quality criterion when several solutions are possible for a certain task in three cushion billiards).

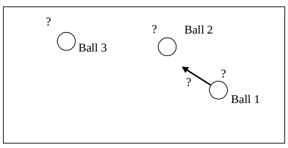

FIGURE 6. Setting up a configuration with certain properties

<u>*Example 3:*</u> *Find a configuration (i.e. the positions of the balls and a velocity vector for ball 1), such that ball 1 hits ball 2 and would hit ball 3 afterwards if the latter had not been moved away by ball 2 in the meantime (see e.g. figure 6).*

The solution of this task requires the variation of positions and of the velocity vector in order to achieve the desired effect. Performing the “experiments” with pencil and paper is quite tedious such that a simulation would be the right tool to use.

<u>Example 4:</u> *Given the (center) positions of a ball rolling on the table before and after hitting a cushion, determine the coefficient of restitution for the collision between a ball and a cushion.*
For performing this task, one can compute lines of regression for the path before and after collision and compute k from the angles (for the formula cf. the model description above).

The learning goals for students working on these tasks can be outlined as follows:

- Students should recognise that basic geometrical concepts and operations like line, circle, reflection etc. are useful in solving practical problems. They should be able to use the concepts and construction methods (orthogonal line, tangent, reflection, dilation etc) to determine positions and directions. They should also recognise that and how geometric theorems like the theorem of Thales can be used for constructive purposes.
- Students should be able to set up corresponding algebraic models and set up and solve (using technology) the algebraic equations. This includes the equations for lines and circles, equations in triangles (rule of sine/cosine, theorem of Pythagoras), two-dimensional coordinate geometry and vectors.
- Students should learn how to tackle a complex problem by starting with special cases and simplified situations and reducing the more complex cases using the results obtained earlier.

These goals are important in secondary as well as tertiary education.

4. BILLARDS IN A DYNAMIC GEOMETRY SYSTEM (DGS)

In our scenario, the DGS can be used as a constructive tool for setting up problem situations and for finding positions and velocity directions. When setting up the problem situation (table, initial positions of balls), interesting questions already come up. It is a quite natural requirement to perform the table construction in such a way that it can be easily adapted to different table sizes. Therefore, it is not sufficient to draw a simple rectangle using four suitable points in a mesh. If one then moves one point, the rectangle property is destroyed. So, the first challenge for students consists of finding a construction using parallel and orthogonal lines such that the rectangle property is invariant when making the table larger. This is an open task with different possible solutions. Such a “design for modification” is also required later on when ball positions and velocity directions are to be varied.
If an exact solution for a problem is not known yet, the method of “soft construction” (Healy 2000) can be applied in order to get ideas. Here, one requirement for an exact construction is left out and the dragging mode of the DGS is used to find a position or angle such that the remaining condition is also fulfilled. This configuration is then investigated in order to detect an additional property which could be used for exact construction (“robust construction” in terms of Healy). Moving a soft construction to achieve an additional property is well-known from classical problems concerning constructible numbers (e.g. trisection of an angle).

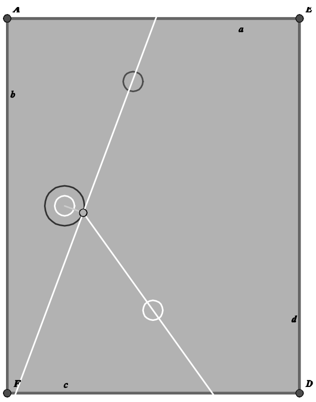

FIGURE 7. Construction of carambolage

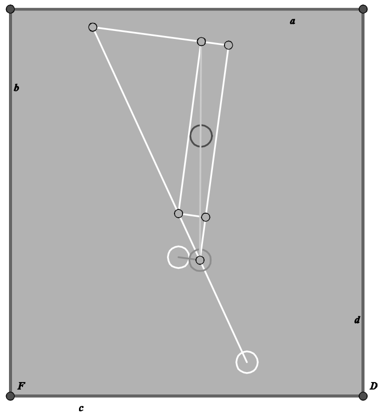

FIGURE 8. Construction with loss of energy

Figure 7 shows a soft construction for a carambolage in the model without loss of kinetic energy. The line through the lowest ball can be rotated and by doing this one can achieve that the second line goes through the center of the third ball. Students then have to retrieve properties from this desired configuration in order to do a “robust construction”. In this case, one might see that the center of the first ball, when colliding with the second one, also lies on the Thales circle over the line segment between the centers of balls 2 and 3. Students can always make use of the Thales theorem in such a constructive way when a point with unknown position is the angular point of a right angle where points on the legs are known. Figure 8 shows a soft construction for the same problem in the model including loss of energy. A construction using the theorem of intersecting lines is used to guarantee the relation between $\tan(\delta)$ and $\tan(\alpha)$.

Soft constructions can be considered as blue prints for real mechanisms, i.e. aiming devices. This way, one gets from modelling reality to realizing models. The usage of DGS for constructing and animating planar mechanisms can also be found in mechanism theory and design (Corves 2004), and we actually realised a Cinderella construction as a coupler device as will be shown in the last section.

When using a DGS for solving problems in the Billiards environment, there are also some limitations. One can only model potential motion lines (or paths). There might be other balls getting in the way or balls might stop because of friction. So, there is no dynamic modelling over several collisions and final resting positions cannot be constructed. Moreover, the construction does not give any feedback on the real motion (“real” with respect to one of the models outlined above) as a simulation does. Hence, one cannot make experiments to get initial ideas on how solutions might look like and one cannot check results.

5. BILLIARDS MACHINE

The real billiards machine is depicted in Figure 9. Balls can be positioned on the table using the 3-axes-machine and a sucking device. A cylinder that is attached to the vertical axis is used as a substitute cue. This way, impacts are always centrical, English shots cannot be performed. Since this is left out in all of our models, this is not an important restriction but rather desired. The cylinder can be programmed to move in a certain direction with a given velocity. Hence it can be used to test constructions performed on paper or with a DGS. We had a camera installed temporarily in order to record real motion from which we computed the coefficient of friction and the coefficients of restitution. It would be helpful to have such a device permanently installed combined with a software which computes the center coordinates of balls in single pictures.

FIGURE 9. Billiards Machine

FIGURE 10. Aiming Device

We see several kinds of usage within the billiards learning scenario:

1. check validity of a model, give rise to model propagation
2. retrieve data for coefficients from camera pictures
3. test and adapt coefficients by trials
4. retrieve data on realistic restrictions (e.g. maximum velocity).

Real objects can show the difference between model and reality and give rise to model progression. Seeing in the real billiards machine, that the law of reflection is violated gives rise to get from model stage 2 to stage 3.

6. DISCUSSION AND CONCLUSIONS

We consider two kinds of reality ("real" billiards, billiards machine) and a sequence of models (mathematical billiards, physical billiards with friction, physical billiards with friction and loss of energy). The billiards machine provides a subset of the "real human billiards" in that it only allows centrically straight shots in the plane. The sequence of models we describe includes more and more features necessary to model the reality of the billiards machine. Thus, it provides a pathway to proceed from problem solving in a simple model to that in a more complex, realistic model.

As yet, we have used the billiards scenario in mathematical application projects as well as in diploma theses for mechanical engineering students (Alpers 2002). In the projects, students worked with the billiards machine reality. They have learnt about the underlying impact situation in their technical mechanics lectures. They recall their knowledge to set up the models presented in this paper, so the modelling part rather consists of understanding already existing models. The main task then is to solve problems within the models and check with reality how well the respective model fits. Solving a problem in a simpler model often gives hints on solving it in more complex models (making changes instead of starting from scratch). Moreover, it is a good engineering principle to keep a model as simple as possible.

One project dealt with the situation described in example 2. Students had to set up a CAS worksheet for computing centrically straight shots via a cushion in the different models. They then realised the shots on the billiards machine which showed that the model without friction produced shots without collision whereas the model including loss of energy led to a shot that was “nearly” centrically straight. Another student group had the task to investigate the model assumption in the most advanced model that there is a constant coefficient of restitution for ball-cushion collisions. They first had to understand the respective model and then to think up a measurement scenario for computing the coefficient k. They performed several trials with the Billiards machine using different “ingoing” angles and measured the “outgoing angles” from which k can be computed. These resulted in values varying around 0.7 (from 0.6 to 0.8) showing that the modelling assumption is reasonable but the possible variation must also be taken into account. In yet another project, position data retrieved from a highspeed camera was used to compute the angles using regression lines. Moreover, for the configuration shown in figure 8 (carambolage in the model including loss of energy), an aiming device was constructed that is depicted in figure 10. For this, the respective DGS construction was used as a blue print for the device. The students had to first understand the construction and then convert it into a coupling structure with swivel and translational joints. The DGS turned out to be an excellent support for understanding the motion of the joints making up the device. We got the overall impression that the Billiards scenario provides a wealth of project tasks for a variety of levels of difficulty and time frames.

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